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A CLASS OF UNBIASED ESTIMATORS OF PRODUCT OF POPULATION MEANS

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SUMMARY

In this paper an almost unbiased estimator of product of population means of two characteristics is introduced and compared with the usual biased and adjusted unbiased estimators. A class of unbiased estimators of product of population means is also considered.

Keywords : Estimators of product, Bias, Mean Square Error.

1. Introduction and Estimator

For a simple random sample of size n, drawn without replacement from a population of size N, let $\bar{x}$ and $\bar{y}$ be the sample mean estimators of μ_x and μ_y' respectively, the population means of the characteristics x and y. The conventional estimator of product of the population means, $P = \mu_y \mu_x$ is $p = \bar{y}\bar{x}$.

The bias of estimator $p = \bar{y}\bar{x}$ is

$$E(\bar{y}\bar{x}) - P = E[\bar{y}\bar{x} - \mu_y \mu_x]$$

$$= P \frac{\text{Cov}(\bar{x}, \bar{y})}{\mu_y \mu_x}$$

$$= P\, C_{xy}$$

From this an adjusted, unbiased estimator of P is

$$t^* = \bar{y}\bar{x}\,[1-\hat{C}_{xy}] \tag{1}$$

or

$$t^* = \bar{y}\bar{x}\left[1-\frac{(1-f)}{n}\cdot\frac{\hat{S}_{xy}}{\bar{x}\bar{y}}\right] \tag{2}$$

where $\hat{C}_{xy}=\dfrac{(1-f)}{n}\,\dfrac{\hat{S}_{xy}}{\bar{x}\bar{y}}$ is the usual consistent estimator of

$$C_{xy}=\frac{\text{Cov}\,(\bar{x},\bar{y})}{\mu_x\mu_y},\ \text{where}\ \hat{S}_{xy}=(n-1)^{-1}\sum_{i=1}^{n}(x_i-\bar{x})\,(y_i-\bar{y})$$

and $f=\dfrac{n}{N}$.

The expression of the bias of the estimator P, gives

$$P=\frac{E\,(\bar{x}\bar{y})}{(1+C_{xy})}$$

We obtain an almost unbiased estimator of P,

$$t_1^* = \frac{\bar{y}\,\bar{x}}{(1+\hat{C}_{xy})}$$

or

$$t_1^* = \bar{y}\bar{x}\Big/\left[1+\frac{(1-f)}{n}\,\frac{\hat{S}_{xy}}{\bar{x}\bar{y}}\right] \tag{3}$$

It is interesting to compare the present method of constructing t_1^* with the technique adopted by Sahoo and Swain [1] and Sahoo [2].

2. Comparison of t_1^* with p and t^*

In this section we shall compare the efficiency of t_1^* with that of p and t^* separately. The relative mean square error (*RMSE*) of t_1^* is obtained by using Taylor's series expansion. Further for computing the necessary moments and product moments of $\bar{y}$, $\bar{x}$ and $\hat{S}_{xy}$, we use the same notations and approximations used by Srivastava, Shukla and Bhatnagar [4]. Thus, the relative mean square error of t_1^* to terms of order $o\,(n^{-2})$ is obtained as

$$RMSE\,(t_1^*)=\frac{(1-f)}{n}\Big[\,C_{20}+C_{02}+2C_{11}+\frac{(1-f)}{n}\Big\{\,C_{02}C_{20}+C_{11}^2 -\frac{2f}{(1-f)}\,(C_{21}+C_{12})\Big\}\Big]$$

$$\text{where}\quad C_{ab}=\frac{1}{(N-1)}\sum_{i=1}^{N}\left(\frac{x_i-\mu_x}{\mu_x}\right)^a\left(\frac{y_i-\mu_y}{\mu_y}\right)^b, \tag{4}$$

(a,b) being the non-negative integers. But this is equal to the relative *MSE* of t^*, showing that the two estimators t^* and t_1^* are equally efficient to terms of order $o(n^{-2})$.

In order to compare t_1^* and t^* with usual product estimator p, we below give the exact relative *MSE* of p as

$$RMSE\,(p)=\frac{(1-f)}{n}\Big[\,C_{02}+C_{20}+2C_{11}+\frac{1}{n}\Big\{\frac{(N^2+N-6nN+6n^2)}{(N-2)\,(N-3)n}\,C_{22} +\frac{(n-1)\,N(N-n-1)}{n(N-2)\,(N-3)}\,(C_{20}C_{02}+2C_{11}^2 +\frac{2(1-2f)N}{(N-2)}\,(C_{21}+C_{12})\Big\}\Big]$$

$$\text{or}\ RMSE\,(p)\cong\frac{(1-f)}{n}\Big[\,C_{02}+C_{20}+2C_{11}+\frac{(1-f)}{n}\Big\{\,C_{20}C_{02}+2C_{11}^2 +\frac{2(1-2f)}{(1-f)}\,(C_{21}+C_{12})\Big\}+\frac{1}{n^2}\,C_{22}\,\Big],$$

Thus, the relative *MSE*, to terms of order $o\,(n^{-2})$, of p is

$$RMSE\,(p)=\frac{(1-f)}{n}\Big[\,C_{02}+C_{20}+2C_{11}+\frac{(1-f)}{n}\Big\{\,C_{20}C_{02}+2C_{11}^2 +\frac{2(1-2f)}{(1-f)}\,(C_{21}+C_{12})\Big\}\Big] \tag{5}$$

which is the same as reported by Srivastava, Shukla and Bhatnagar [4], equation (2.4), p.192).

The relative *MSE* of t_1^* in (10) can be re-written as

$$RMSE\,(t_1^*)=RMSE\,(t^*)=RMSE\,(p)-\frac{(1-f)^2}{n^2}\Big[C_{11}^2+2\,(C_{21}+C_{12})\Big], \tag{6}$$

It follows from (6) that the unbiased/almost unbiased estimators t^* and t_1^* are more efficient than usual estimator p (to terms of order $o\,(n^{-2})$ when

$$[C_{11}^2+2\,(C_{21}+C_{12})]\geqslant o \tag{7}$$

The condition (7) is satisfied for symmetric bivariate distributions of x and y for which $C_{21}=C_{12}=o$.

A choice between the two estimators t^* and t_1^* depends on the comparison of the relative mean square error to terms of order $o\,(n^{-3})$. Unfortunately, the results on the relative mean square errors to this order of approximation are too complicated to lead to useful guide for practical purposes. If we assume that the population is infinite and the joint distribution of x and y is bivariate normal, the results simplify considerably and we get the relative mean square errors of p, t^* and t_1^*, respectively as

$$RMSE\,(p)=\frac{1}{n}\Big[C_{02}+C_{20}+2C_{11}+\frac{1}{n}\Big(1+\frac{1}{n}\Big)(1+2\rho^2)\,C_{02}C_{20}\Big] \tag{8}$$

$$RMSE\,(t^*)=\frac{1}{n}\Big[C_{20}+C_{02}+2C_{11}+\frac{1}{n}\Big\{1+\rho^2+\frac{1}{n}(1+2\rho^2)\Big\}C_{02}C_{20}\Big], \tag{9}$$

and $$RMSE\,(t_1^*)=RMSE\,(t^*)-\frac{2C_{11}^2}{n^3}\,(C_{02}+C_{20}+2C_{11}), \tag{10}$$

where ρ is the correlation coefficient between x and y. We have from (8), (9) and (10) that

$$RMSE\,(p)-RMSE\,(t^*)=\rho^2\,C_{02}C_{20}>0, \tag{11}$$

and

$$RMSE\,(t^*) - RMSE\,(t_1^*) = \frac{2C_{11}^2}{n^3}\,(C_{02}+C_{20}+2C_{11}) \tag{12}$$

$$>0$$

Thus we have the following inequality :

$$RMSE\,(t_1^*) \leqslant RMSE\,(t^*) \leqslant RMSE\,(p) \tag{13}$$

which follows that the proposed estimator t_1^* is more efficient than each p and t^*.

3. A Class of Estimators

Whatever be the sample chosen let $t=\hat{C}_{xy}$ assume values in a bounded, closed convex subset, S, of the one dimensional real space containing the point $T=C_{xy}$. Let $g\,(t)$ be a function of t (which in particular may be a polynomial in $\hat{C}_{xy}$) satisfying the following conditions :

1. The function $g\,(t)$ is continuous and bounded in S,
2. The first and second order partial derivatives of $g\,(t)$ exist and are continuous and bounded in S.
3. After expansion under the given conditions

$$g\,(t) = 1 - t + \sum_{r=2}^{\infty} a_r\, t^r \quad \text{(terms involving second and higher powers of } t) \tag{14}$$

Hence, we have the following theorem :

Theorem : Let $g\,(t)$ be a function of $t=\hat{C}_{xy}$ such that it satisfies the above regularity conditions. Then, a class of almost unbiased estimators of product p may be defined by

$$d_g = p\, g\,(t) \tag{15}$$

Such a class of estimators has been discussed by Sahoo [3] for ratio of population means of two characters x and y.

Since there are only a finite number of samples, the bias and mean square error of the estimator d_g exist under the conditions 1 and 3. Also we note that the bias of d_g is of order n^{-2}, so that its contribution to the mean square error will be of order n^{-4}. This shows that to terms of order n^{-2} all the estimators of the class (15) have the same mean square error.

d_g generates a class of almost unbiased estimators of P by substituting proper choice of $g(t)$. It may be seen that the estimators t^*, t_1^* and the estimators like

$$P_1 = p\, e^{-\hat{C}_{xy}}$$

$$P_2 = p\, (1-\hat{C}_{xy})^{1/2}\, (1+\hat{C}_{xy})^{-1/2}$$

$$P_3 = p\, (1-\hat{C}_{xy})^{1/4}\, (1+\hat{C}_{xy})^{-3/4}$$

$$P_4 = p\, (1-\hat{C}_{xy})^{\alpha}\, (1+\hat{C}_{yx})^{-\beta} \text{ etc.}$$

such that $\alpha+\beta=1$, are the members of the class d_g.

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